From Unique Factorization Domains (UFDs) To Unique Factorization Modules (UFMs): A Survey.

Sri Wahyuni[1], Hidetoshi Marubayashi[2]

[1]Dept. of Mathematics, Universitas Gadjah Mada, Yogyakarta - Indonesia.
swahyuni@ugm.ac.id
[2]Dept. of Mathematics, Naruto University of Education, Japan.
marubaya@naruto-u.ac.jp

Presented at The Online ICW-HDDA-X Meeting
October 14, 2020

The Presentation:

1. Some Preliminary Notions
2. Unique Factorization Module
3. Characterization of UFM when D is a Krull Domain
4. Characterization of UFM when D is an Integrally Closed Domain (current work)
5. Some Open Problems : In connection between D-module M and $D[x]$-module $M[x]$.

Some Preliminary Notions: R-module M

Let

- Some Preliminary Notions
- $(R, +, \cdot)$ be a ring with unity (say 1_R)
- $\cdot : R \times M \to M$ be an operation, and noted by $\cdot(r, m) = r \cdot m$

M is called **an R-module** if

1. $(r_1 + r_2).m_1 = r_1.m_1 + r_2.m_1$
2. $r_1.(m_1 + m_2) = r_1.m_1 + r_1.m_2$
3. $r_1.r_2.m_1 = r_1.(r_2.m_1)$
4. $1_R.m_1 = m_1$

$\forall\ r_1, r_2 \in R$ and $\forall\ m_1, m_2 \in M$.

The Idea comes from the fact every ring R is a module over R (him/her-self)

- Let R be a ring

$$(R, +, \dot{}) : ring \quad with \quad unity \quad \Rightarrow \quad R \quad is \quad R - module.$$

- A General natural question: **is a concept on ring could be shifted to concept on module?**
- Including the concept of **Unique Factorization Domains (UFD)**, can we shift to the concept of **Unique Factorization Modules (UFM)?**

$Q(D)$ module over D.

- Let D be an integral domain
- Define equivalent relation $\sim$ on $D \times (D - \{0\})$ by

$$(p_1, q_1) \sim (p_2, q_2) \Leftrightarrow p_1 q_2 = p_2 q_1,$$

  $(\forall (p_1, q_1), (p_2, q_2) \in D \times (D - \{0\})$.
- Let $Q(D)$ be set of all equivalent classes due to $\sim$ on $D \times (D - \{0\})$

$$Q(D) = \{\frac{p}{q} \mid (p, q) \in D \times D - \{0\}\}$$

  $\frac{p}{q}$ is a notation of equivalent class represented by (p, q).
- $(Q(D), +)$: **an abelian group** with $(+)$ defined as follows:

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}, \qquad (\forall \frac{p_1}{q_1}, \frac{p_2}{q_2} \in Q(D))$$

$Q(D)$ module over D.

- Define $\cdot : D \times Q(D) \to Q(D)$ as follows:

$$d \cdot \frac{p}{q} = \frac{dp}{q}$$

  $(\forall d \in D, \frac{p}{q} \in Q(D))$.

- Then we have $Q(D)$ as **module over** D.

Fractional D-ideal

Let D be a commutative domain with its quotient field $K = Q(D)$. Using D-module K we define **Fractional D-ideal**

Definition

- Let $\mathfrak{a}$ be a non zero submodule of module K over D.
- A submodule $\mathfrak{a}$ is called a **fractional D-ideal** if there exist a non-zero $r \in D$ such that $r\mathfrak{a} \subseteq D$.
- $\mathfrak{a}$ is called **invertible** if $\mathfrak{a}^{-1}\mathfrak{a} = D$.
- Notation $\mathfrak{a}^{-1} = (D : \mathfrak{a}) = \{k \in K \mid k\mathfrak{a} \subseteq D\}$.

- If $D \supseteq \mathfrak{a}$, then

  $\mathfrak{a}$ is a fractional ideal if and only if it is non-zero ideal.

Dedekind Domain

Theorem

The following conditions on D are equivalent:

- *Every ideal of D is invertible*
- *Every fractional D-ideal is invertible*
- *The family $F(D)$ of all fractional D-ideals is a group under usual ideal product*
- *Every non-zero ideal of D is a finite product of maximal ideals.*

Definition

D is called a **Dedekind Domain** if D satisfies one on the conditions of above theorem.

Dedekind Module and Multiplication Module

Definition

- M is called a **Dedekind Module** if every non-zero submodule N of M is invertible, that is $N^{-}N = M$, where $N^{-} = \{k \in K \mid kN \subseteq M\}$.
- M is called a **multiplication module** if for each submodule N of M there is an ideal $\mathfrak{n}$ of D such that $N = \mathfrak{n}M$.

Note:

- **Compare to the definition of Dedekind Domains defined before**
- **It is clear that the definition of Dedekind Module is an extension of the definition of Dedekind Domains.**

v-operation and fractional v-ideal, and defintion of UFD.

Definition

- v-**operation**: For a fractional D-ideal $\mathfrak{a}$, we define a set

$$\mathfrak{a}_v = (D : (D : A))$$

- Then we will have $\mathfrak{a}_v \supseteq \mathfrak{a}$.
- Not always happens $\mathfrak{a}_v = \mathfrak{a}$.
- $\mathfrak{a}$ is called **(fractional)** v-**ideal** if $\mathfrak{a} = \mathfrak{a}_v$.

Definition

D is called a **Unique Factorization Domain (UFD)** if every non-unit element in D is a finite product of prime element (by prime element p we mean pD is a prime ideal).

Characterization of UFD

Theorem (Marubayashi (2018))

The following conditions on D are equivalent:

- *D is a UFD*
- *For every non unit element a in D, aD is a finite product of a principal prime ideals.*
- *Every prime v-ideal P is a principal, that is, $P = pD$ for some $p \in P$(Samuel, 1964).*
- *Every non-zero prime ideal contains a prime element (Kaplansky, 1974)*

Definition

D is called a **Generalized Dedekind** (a G-Dedekind Domain) if

1. Every v-ideal D is invertible.
2. D satisfies the ascending chain conditions on v-ideals of D.

v-ideal and Krull Domain

For a fractional D-ideal $\mathfrak{a}$, we define $\mathfrak{a}_v = (D : (D : \mathfrak{a}))$. It is clear that $\mathfrak{a} \subseteq \mathfrak{a}_v$. If $\mathfrak{a} = \mathfrak{a}_v$, then $\mathfrak{a}$ is called a *fractional v-ideal*. Further, $\mathfrak{a}$ is said to be *v-invertible* if $(\mathfrak{a}^{-1}\mathfrak{a})_v = D$.

Definition

D is called a **Krull domain** if:

1. Every v-ideal of D is v-invertible.
2. D satisfies the ascending chain condition on v-ideals of D (D is v-Noetherian).

Characterization of a Krull domain using the concept of v-ideals

From the above definitions, we conclude that every G-Dedekind domain is a Krull domain. The following theorem will give us the characterization of a Krull domain using the concept of v-ideals.

Theorem (Marubayashi (2018))

The following conditions on D are equivalent:

a. *D is a Krull domain.*

b.
   1. *The set $G_v(D)$ of all fractional v-ideals is a group under the v-product "$\circ$", that is $\mathfrak{a} \circ \mathfrak{b} = (\mathfrak{a}\mathfrak{b})_v$ for all fractional v-ideals $\mathfrak{a}$ and $\mathfrak{b}$.*
   2. *Any v-ideals of D is a finite v-product of maximal v-ideals (maximal ideal amongst the v-ideals of D).*

Proof.

Theorem 4.7 in (Marubayashi (2018)). □

Theorem

([Marubayashi (2018)]) *The following conditions on D is equivalent:*

1. *D is a UFD.*
2. *For every non-unit element $a \in D$, aD is a finite product of principal prime ideals.*
3. *Every prime v-ideal P is principal, that is, $P = pD$ for some $p \in P$.* **Samuel ([16])**
4. *Every non-zero prime ideal contains a prime element.* **Kaplansky ([9])**

Note:

- In domain theory, we have the following relation

  R : PID $\Longrightarrow$ R UFD $\Longrightarrow$ R: G-Dedekind domain $\Longrightarrow$ R: Krull domain.

- And we know that D is UFD if and only if so is $D[x]$.
- We are studying the concept of extending unique factorization domains to unique factorization module.

Fractional D-submodule as a tool to define Unique Factorization Module

If we assume M is a finitely generated torsion-free D module, then $M \subset KM$ and $KM \cong K \oplus \cdots \oplus K$, n-copies of K.

Definition

1. A D-submodule N of KM is called a **fractional D-submodule in KM** if there is a $0 \neq r \in D$ such that $rN \subseteq M$ and $KN = KM$.
2. A D-submodule $\mathfrak{a}$ of K is called a **fractional M-ideal** if there is a $0 \neq m \in M$ such that $\mathfrak{a}m \subseteq M$ and $K\mathfrak{a}^+ = KM$, where $\mathfrak{a}^+ = \{m' \in KM \mid \mathfrak{a}m' \subseteq M\}$, a fractional D-submodule.

Note:
Compare to the definition of Fractional D-ideal defined before.

Fractional D-submodule as a tool to define Unique Factorization Module

Definition

1. If N is a fractional D-submodule in KM, we define $N^{-} = \{k \in K \mid kN \subseteq M\}$, a fractional M-ideal in K, and $N_v = (N^{-})^{+}$, a fractional D-submodule in KM. And we have $N_v \supseteq N$, not always $N_v = N$
2. A fractional D-submodule N is called a v-**submodule** in KM if $N_v = N$.

Unique Factorization Module (UFM)

Definition

Let M be a finitely generated torsion-free D-module and let D be an integrally closed domain. M is a **unique factorization module** (UFM) if the following two conditions are satisfied:

1. every v-submodule N of M is principal, that is $N = rM$ for some $r \in D$.
2. M satisfies the ascending chain condition on v-submodules of M.

Note: Compare to the definition of Unique Factorization Domain (UFD)

Other version of the definition of Unique Factorization Module (UFM)

- There are some other versions of the definition of unique factorization module with different approach (see Alan and Özbulur (2016) [1], Fletcher (1969) [8], Rouenteen and Namazi (2011) [15]), we do not say our definition is best or not?
- Let us see the approach by Rouenteen and Namazi (2011) [15]

Starting from integral module as a generalization of integral domain: [15] approach

Definition (Roueentan-Namazi, 2011)

An R-module M is called **integral module** if $M = 0$ or its zero submodule is prime is prime M, the relation $rm = 0$ for $m \in M$, $r \in R$ implies that $m = 0$ or $rM = 0$ ($r \in ann(M)$).

It is easy to check that every simple module is an integral module. Also if M is an R-module and N is a proper submodule of M, then N is a prime submodule of M if and only if $M/_N$ is an integral R-module.

Lemma (Roueentan-Namazi, 2011)

Let M be a non-zero multiplication R-module then M is an integral module if and only if $mm' = 0$ implies that $m = 0$ or $m' = 0$ $(m, m' \in M)$.

Divisibility and associated relation.

Definition (Roueentan-Namazi, 2011)

Let M be an R-module. If N and K are submodules of M.

- We say that N divides K if there exists an ideal I of R such that $K = IN$, denoted by $N \mid K$.
- In particular for elements m and m' of M we say that m divides m' if $Rm' \mid Rm$. In this case we say that m' is a divisor of m and we write $m' \mid m$. Clearly $m' \mid m$ if and only if $Rm \subseteq Rm'$.
- Elements m', m of M are said to be associated if $m' \mid m$ and $m \mid m'$.

Definition (Roueentan-Namazi, 2011)

Let $0 \neq m$ be an element of a multiplication R-module M.

- $0 \neq u \in M$ is called **unit** if $Ru = M$.
- We say that $m \in M$ is irreducible provided that: (1) m is non-unit, and (2) If $Rm = R(sm')$, $s \in R$ and $m' \in M$, then either m is a unit in M or $s + ann(M)$ is a unit in $R_{ann(M)}$.
- We say that m is prime provided that $N = Rm$ is a prime submodule of M.

Definition of UFM by Roueentan-Namazi (2011)

Definition (Roueentan-Namazi, 2011)

If M is a multiplication integral R-module, then we say that M is a unique factorization module provided that:

1. for each non-zero non-unit element m of M, the submodule Rm can be written as $Rm = (Rm_1)\cdots(Rm_n)$, where $m_1, \ldots, m_n$ irreducible in M.
2. If $Rm = (Rm'_1)\cdots(Rm'_n) = (Rm_1)\cdots(Rm_n); m_i, m'_i$ irreducible), then $n = r$ and for each i, m_i and m'_i are associates, apart from the order in which the factors occur.

For example every simple module is a unique factorization module. Also every cyclic module over a principal integral domain is principal.

Theorem (Roueentan-Namazi, 2011)

Every principal integral module is a unique factorization module.

v-multiplication module

- For a fractional D-submodule N in KM, we introduce the notion $N^{-} = \{k \in K \mid kN \subseteq M\}$, a fractional M-ideal in K. If $N^{-}N = M$, then N is said to be *invertible.* Now we will introduce the following v-operation.
- First, we define the notion $N_v = (N^{-})^{+}$, a fractional D-submodule in KM. As in the domain case, it is clear that $N \subseteq N_v$. If $N = N_v$, then N is said to be a *v-submodule* in KM.

Definition

A D-module M is called a *v-multiplication module* if each v-submodule of M is a multiplication submodule.

G-Dedekind Module

Motivated by the definition of **G-Dedekind Domain**, we define the definition of **G-Dedekind Module** as follows.

Definition

M is called a **G-Dedekind Module** if

1. every v-submodule of M is invertible
2. M satisfies the ascending chain condition on v-submodules of M.

G-Dedekind Module

Further we have the following theorem.

Theorem

Wijayanti at.al (2019) [10]*Suppose D is a Dedekind domian and M is finitely torsion-free D-module. Then*

1. *M is a G-Dedekind module.*
2. *Every v-submodule N of M is of the form:* $N = \mathfrak{n}M$, *where* $\mathfrak{n} = (N : M) = \{r \in D \mid rM \subseteq N\}$.

Moreover, motivated by the definition of **Krull Domain**, we define the definition of **Krull Module** as follows.

Definition

M is called a **Krull Module** if

1. every v-submodule N of M is v-invertible, that is $(N^{-}N)_v = M$
2. M satisfies the ascending chain condition on v-submodules of M.

Characterization of UFM when D is a Krull Domain

Theorem

Let D be a Krull domain. Then the following conditions on M are equivalent:

(1) *M is a UFM,*

(2) *M is a v-multiplication module and D is a UFD,*

(3) *Every prime v-submodule is principal and M satisfies the ACC on v-submodules of M,*

(4) *Every v-submodule is principal.*

Proof:

Proof.

- (1) $\Longrightarrow$ (3): It is clear from the definition that every v-submodule of M is principal. So are the prime v-submodules of M.
- (1) $\Longrightarrow$ (4): Obvious by the definition.
- (4) $\Longrightarrow$ (1): Let every v-submodule in M is principal. To show that M is UFM, it is enough to show M satisfies the ACC on v-submodules of M. Using the assumption D be a Krull domain then follow result on Ernanto et.al. [11].
- (1) $\Longrightarrow$ (2): Let M be a UFM. It clear every v-submodule N of M is principal, that is $N = rM$ for some $r \in D$, which means every v-submodule is multiplication. That D is UFD, it follows from the assumption that D is Krull domains.
- (2) $\Longrightarrow$ (3): Let M is a v-multiplication module and D is a UFD. Then every v-submodule is a multiplication submodule on M, including the prime v-submodule which means every prime v-submodule is principal. The same above argument the ACC will follow from $M = \mathfrak{n}^{-1}\mathfrak{n}M$ and

Corollary

Corollary

If M is a projective D-module and D is a UFD, Then M is a UFM.

If D is a G-Dedekind domain with its quotient field K such that D satisfies the ascending chain condition on v-ideals of D and M is a finitely generated torsion free D-module, then we have the following theorem.

Theorem

Suppose M is a projective D-module. Then M is a G-Dedekind module and is a v-multiplication module.

Proof:

First we prove that an integral D-submodule P is a maximal v-submodule (submodules maximal amongst the v-submodules) if and only if $P = \mathfrak{p}M$ for some prime v-ideal $\mathfrak{p}$ of D.
Let $\mathfrak{p}$ be a prime v-ideal of D. Then it is invertible. Put $P = \mathfrak{p}M$. Then P is a v-submodule by Wijayanti et.al. [[10], Lemmas 2.4 and 3.2] and $P^- = \mathfrak{p}^{-1}$.
1. In case M is a free D-module, that is, $M = D \oplus \cdots \oplus D$ (finite copies of D). Let L be a v-submodule of M with $M \supset L \supseteq P$. Then $\mathfrak{p}^{-1} = P^- \supseteq L^- \supset D$ and $D \supseteq \mathfrak{p}L^- \supseteq \mathfrak{p}$.Thus $L^- = \mathfrak{p}^{-1}\mathfrak{a}$ for some ideal $\mathfrak{a}$ of D with $D \supseteq \mathfrak{a} \supseteq \mathfrak{p}$. If $\mathfrak{p} = \mathfrak{a}$, then $L^- = D$ and so $L = L_v = (L^-)^+ = D^+ = M$, a contradiction. If $\mathfrak{a} = D$, then $L^- = \mathfrak{p}^{-1}$ and $L = L_v = (\mathfrak{p}^{-1})^+ = \mathfrak{p}M$. If $D \supset \mathfrak{a} \supset \mathfrak{p}$, then $\mathfrak{a}_v = D$. Let $m' \in \mathfrak{a}^+$ and write $m' = k_1 + \cdots + k_n \in KM = K \oplus \cdots \oplus K$. $\mathfrak{a}m' \subseteq M$ implies $\mathfrak{a}k_i \subseteq D$ and so $k_i \in D$ for all i, that is, $\mathfrak{a}^+ = M$. Thus $L = L_v = (\mathfrak{p}^{-1}\mathfrak{a})^+ = \mathfrak{p}\mathfrak{a}^+ = \mathfrak{p}M$ and hence $P = \mathfrak{p}M$ is a maximal v-submodule.

Proof:(*continued*)

In case M is not a free D-module. Put $F = D \oplus \cdots \oplus D = M \oplus M_1$ for some submodule M_1 of F. Put $P_1 = \mathfrak{p}M \oplus \mathfrak{p}M_1 = \mathfrak{p}(M \oplus M_1) = \mathfrak{p}F$. So, by (i), P_1 is a maximal v-submodule of F. Suppose L is a v-submodule of M containing P. Then as (i), $L^- = \mathfrak{p}^{-1}\mathfrak{a}$ for some ideal $\mathfrak{a}$ of D with $D \supset \mathfrak{a} \supset \mathfrak{p}$. Put $L_1 = L \oplus \mathfrak{p}M_1$, a submodule of F. Then $(L_1)^- = L^- \cap (\mathfrak{p}M_1)^- = \mathfrak{p}^{-1}\mathfrak{a} \cap \mathfrak{p}^{-1} = \mathfrak{p}^{-1}\mathfrak{a}$ and $L_1 \subseteq (L_1)_v = (\mathfrak{p}^{-1}\mathfrak{a})^+ = \mathfrak{p}\mathfrak{a}^+ = \mathfrak{p}F = \mathfrak{p}M \oplus \mathfrak{p}M_1$, which shows $L = \mathfrak{p}M$. Hence $P = \mathfrak{p}M$ is a maximal v-submodule for each prime v-ideal of D.

Proof:

Conversely, let P be a maximal v-submodule. Then P is a prime submodule and $\mathfrak{p} = (P : M)$ is a prime v-ideal of D with $P \supseteq \mathfrak{p}M$. Hence $P = \mathfrak{p}M$ follows. Let N be a v-submodule with $\mathfrak{n} = (N : M)$. Then $N \supseteq \mathfrak{n}M$ and $\mathfrak{n}$ is a v-ideal of D. Since D is a G-Dedekind domain $\mathfrak{n} = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}^{e_r}$ for some prime invertible ideals $\mathfrak{p}_i (1 \leq i \leq r)$. Put $n = e_1 + \cdots + e_r$. If $n = 1$, then $\mathfrak{n} = \mathfrak{p}_1$ and $N \supseteq \mathfrak{n}M = \mathfrak{p}_1 M$. Hence $N = \mathfrak{n}M$. We prove that $N = \mathfrak{n}M$ by induction on n. As before, $N^- = \mathfrak{n}^{-1}\mathfrak{a}$ for some ideal $\mathfrak{a}$ of D such that $D \supseteq \mathfrak{a} \supset \mathfrak{n}$. If $\mathfrak{a} = D$, then $N^- = \mathfrak{n}^{-1}$ and so $N = (N^-)^+ = (\mathfrak{n}^{-1})^+ = \mathfrak{n}M$.Thus we may assume that $D \supset \mathfrak{a} \supset \mathfrak{n}$. Put $P_i = \mathfrak{p}_i M (1 \leq i \leq r)$, maximal v-submodules. Suppose $P_i \not\supseteq N$ for all i. Then $M = (P_i + N)_v$ and and $D = M^- = ((P_i + N)_v)^- = (P_i + N)^- = P_i^- \cap N^- = \mathfrak{p}_i^{-1} \cap \mathfrak{n}^{-1}\mathfrak{a}$. It follows that $D_{\mathfrak{p}_i} = (\mathfrak{p}_i^{-1} \cap \mathfrak{n}^{-1}\mathfrak{a})_{\mathfrak{p}_i}$ $=\mathfrak{p}_i^{-1} D_{\mathfrak{p}_i} \cap \mathfrak{n}^{-1} D_{\mathfrak{p}_i} \mathfrak{a}_{\mathfrak{p}_i}$ $=\mathfrak{p}_i^{-1} D_{\mathfrak{p}_i} \cap \mathfrak{p}_i^{-e_i} \mathfrak{a}_{\mathfrak{p}_i}$. If $\mathfrak{a}D_{\mathfrak{p}_i} = \mathfrak{n}D_{\mathfrak{p}_i} = \mathfrak{p}_i^{e_i} D_{\mathfrak{p}_i}$ for all i, then $\mathfrak{a} \subseteq \mathfrak{a}D_{\mathfrak{p}_i} \cap D = \mathfrak{p}_i^{e_i} D_{\mathfrak{p}_i} \cap D = \mathfrak{p}_i^{e_i}$ and $\mathfrak{a} \subseteq \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_r^{e_r} = \mathfrak{n}$, which is a contradiction.

Proof:

Thus there is an i, say, $i = 1$ such that $\mathfrak{a}D_{\mathfrak{p}_1} \supset \mathfrak{n}D_{\mathfrak{p}_1}$. Then $\mathfrak{a}D_{\mathfrak{p}_1} = \mathfrak{p}_1^k D_{\mathfrak{p}_1}$ for some $k(e_1 > k \geq 0)$ since $D_{\mathfrak{p}_1}$ is a discrete rank one valuation domain. Thus $D_{\mathfrak{p}_1} = \mathfrak{p}_1^{-1} \cap \mathfrak{p}_1^{k-e_1} D_{\mathfrak{p}_1} = \mathfrak{p}_1^{-1} D_{\mathfrak{p}_1}$, a contradiction. Hence there is a j such that $P_j \supseteq N$ and we may assume that $j = 1$ again. Then $M \supseteq \mathfrak{p}_1^{-1} N \supseteq N$ and $\mathfrak{p}_1^{-1} N$ is a v-submodule. It follows that $(\mathfrak{p}_1^{-1} N : M) = \mathfrak{p}_1^{-1}\mathfrak{n} = \mathfrak{p}_1^{e-1}\mathfrak{p}_2^{e_2} \ldots \mathfrak{p}_r^{e_r}$ and by induction on n we have $\mathfrak{p}_1^{-1} N = \mathfrak{p}_1^{-1}\mathfrak{n}M$. Hence $N = \mathfrak{n}M$ as desired. Now it is easy to see that M satisfies the ascending chain condition on v-submodules of M. Therefore M is a G-Dedekind module and is a v-multiplication module.

The current work: Characterization of UFM when D is an Integrally Closed Domain

Now we are working on the situation when D is an integrally closed domain.
Let D is an integrally closed domain that is an integral domain whose integral closure in its field of fractions is D itself. We have the following conjectures which still some further detail investigations:

Theorem

Let D be D is an Integrally Closed Domain. Then the following conditions on M are equivalent:

(1) *M is a UFM,*

(2) *M is a v-multiplication module and D is a UFD,*

(3) *(i). D is UFD and (ii). for every principal prime ideal $\mathfrak{p}$ of D, $P = \mathfrak{p}M$ is a maximal v-submodule of M.*

(4) *Every v-submodule of M is principal and D is UFD .*

(5) *(i). Every v-submodule of M is principal and (ii). M satisfies the ACC on v-submodules of M.*

Connection between D-module M and $D[x]$-module $M[x]$

When we have a D-module M,

- Ring of polynomial $D[x] = \{p(x) = \sum_{i=0}^{n} d_i x^i \mid d_i \in D\}$, and
- Group Abelian $M[x] = \{m(x) = \sum_{j=0}^{k} m_j x^j \mid m_j \in M\}$, and
- Operation $\cdot : D[x] \times M[x] \to M[x]$ with the definition

$$p(x).m(x) = (\sum_{i=0}^{n} d_i x^i). \sum_{j=0}^{k} m_j x^j = \sum_{s=0}^{n+k} m'_s x^s$$

with $m'_s = \sum_{i+j=s} d_i m_j$

it can be shown that $M[x]$ is a module over $D[x]$.

Open question: If M is a UFM over D, then so is $M[x]$ over $D[x]$?

Acknowledgement:

1. This work is supported by grant **"Hibah Penelitian Departemen Matematika FMIPA UGM 2020"** and **WCP Project Dikti 2019.**
2. The authors thank to **the Organizer of ICW-HDDA-X-2020**, for inviting me to share this work at this very important meeting.

The References:

- M. Alan and E. Özbulur, On Unique Factorization Modules, *Int. Journal of Pure and Applied Mathematics*, **108** (1), pp. 23-28 (2016)
- M. M. Ali and D. J. Smith, Generalized GCD modules, *Beitrage zur Algebra und Geometrie*, **46** (2), pp. 447-466 (2005)
- R. Ameri, On the prime submodules of multiplication modules, *International Journal of Mathematics and Mathematical Sciences*, **27**, pp. 1715-1724 (2003)
- M. F. Atiyah and I. G. MacDonald, Introduction to commutative algebra, *Addison-Wesley*, **27**, pp. 1715-1724 (1969)
- M. Behboodi, O. A. S. Karamzadeh, and H. Koohy, Modules whose certain submodules are prime, *Vietnam Journal of Mathematics*, **32** (3), pp. 303-317 (2004)
- Z. A. El-Bast and P. F. Smith, Multiplication modules, *Comm. Algebra*, **16**, (4), pp. 755-779 (1988)

- I. Ernanto, H. Marubayashi, A. Ueda, and S. Wahyuni, Positively graded rings which are unique factorization rings, *Int. Electronic Journal of Algebra*, (submitted, 2019)
- C. R. Fletcher, Unique factorization rings, *Proc. Camb. Phil. Soc.* **65**, pp. 579-583 (1969)
- I. Kaplansky, Commutative Rings, The University of Chicago Press, 1974.
- E.W. Wijayanti, H. Marubayashi, I. Ernanto, Sutopo, Finitely Generated Torsion-Free Modules Over Integrally Closed Domains, *Communications in Algebra*,(submitted 2019).
- I. Ernanto, H. Marubayashi, Arithmetic modules over generalized Dedekind domains, *submitted 2019*
- H. Marubayashi, Introduction to Multiplicative Ideal Theory in Commutative Domains, (A Lecture Note in Universitas Gadjah Mada), 2018.
- H. Marubayashi, Arithmetic Modules over Commutative Domains (A Lecture Note in Universitas Gadjah Mada), 2019.

- A. G. Naoum and F. H. Al-Awan, Dedekind modules, *Comm. in Algebra*, **24**, pp. 397-412 (1996)
- S. Roueentan and S. Namazi, A Theorem on Unique Factorization Domains Analogue for Modules, *Int. J. Contemp. Math. Sciences*, **6** (32), pp. 1589-1596 (2011)
- P. Samuel, Lectures on Unique Factorization Domains, Tata Institute of Fundamental Research, Bombay, 1964.