Sufficient conditions of the equivalency between the Kunugi integral and the Henstock-Kurzweil integral

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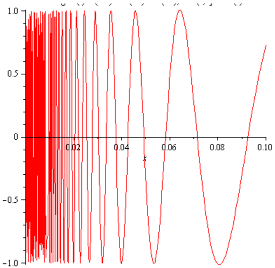
Outline

Abstract

The Kunugi integral is defined by using a fundamental sequence. It is not a simple definition. The Kunugi integral is equivalent with the DL-integral. By using the equivalency, it will be given some sufficient conditions such that the Kunugi integrability of a function on a compact interval is integrable on its every sub compact interval. Based on the results, it will be analyzed some sufficient conditions such that the Kunugi integral and the Henstock-Kurzweil integral are equivalent.

Motivation (1)

The existence of a function which is not Lebesgue integrable function has encouraged many researches to solve the problem by extending the existing integral (Lebesgue integral) to have more general integrals, such as Arnaud Denjoy (1912), Oscar Perron (1914), Bruckner (1986) [Bruckner, 1986], etc.



$f(x) = 2x \sin \frac{2}{x} - \cos \frac{1}{x}$, $x \in (0, 1]$ and $f(0) = 0$.

Motivation (2)

In 1956, Kunugi defined an integral which is familiar as the ER-integral or the Kunugi integral [Kunugi, 1956].

The Kunugi integral on $[a, b]$ is more general than the Lebesgue integral on $[a, b]$ [Kunugi, 1959].

Motivation (3)

Lemma 1

If the functions f and g are Kunugi integrable on $[a, b]$, then

(i) αf *is Kunugi integrable for every* $\alpha \in \mathbb{R}$.

(ii) $f + g$ *is Kunugi integrable on* $[a, b]$.

Motivation (4)

Fact 1: If a functions f is Kunugi integrable on $[a, b]$, there is no guarantee that f is integrable on every $[c, d] \subset [a, b]$ [Lee, 1989].

This is not a usual characteristic of the integral. It is interesting to know more about the **Fact 1**.

Motivation (5)

At the same year of the works of Kunugi, it has been generalized the Riemann integral to be the Henstock-Kurzweil integral.

The Henstock-Kurzweil integral is a non-absoulute integral which is more general than the Lebesgue integral.

It is a generalization of the Riemann integral [Lee, 1989].

The constant δ in the Riemann integral has been generalized to be a positive function $\delta : [a, b] \to \mathbb{R}$.

Motivation (6)

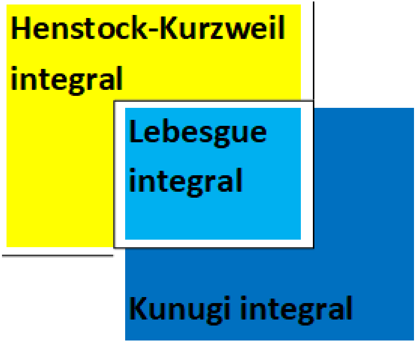
However, it is more enjoyable using a positive constant δ rather than a positive function δ.

A research using a positive constant δ has produced a Riemann-Lebesgue integral (RL-integral) which is equivalent with the absolutely Henstock-Kurzweil integral.

The idea of the RL-integral has been applied to extend the Lebesgue integral. This generalization has given a result an integral which is called by a DL-integral, stands for the Denjoy-Lebesgue integral.

Motivation(7)

Fact 2:



Motivation(8)

The following function is Kunugi integrable on $[a, b]$, but it is not HK-integrable on $[a, b]$.

Let consider the function $f : [\frac{-1}{2}, \frac{1}{2}] \to \mathbb{R}$, where

$$f(x) = \begin{cases} \dfrac{-1}{x \ln x}, & -\frac{1}{2} \leq x < 0 \text{ or } 0 < x \leq \frac{1}{2} \\ 0, & x = 0. \end{cases} \tag{1}$$

The function f is not HK-integrable on $[0, \frac{1}{2}]$, so it is not HK-integrable on $[\frac{-1}{2}, \frac{1}{2}]$.

Motivation(9)

The following function is HK-integrable on $[a, b]$, but it is not Kunugi integrable on $[a, b]$.

Let consider the function $f : [0, 1] \to \mathbb{R}$, where

$$f(x) = \begin{cases} n, & x \in (\frac{1}{2}\frac{1}{n+1} + \frac{1}{n}, \frac{1}{n}) \\ -n, & x \in (\frac{1}{n+1}, \frac{1}{2}\frac{1}{n+1} + \frac{1}{n}) \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Motivation (10)

According to Brucker [Bruckner, 1986], there exists an integral which is more general than the Lebesgue integral. Indrati [Indrati, 2019] also constructed an integral which is more general than the Lebesgue integral based on countably Lipschitz condition, The proof has been done by using globally small Riemann sums property which is developed from locally small Riemann sums property [Indrati, 2019].

Is there any function which is not Lebesgue integrable, but it is still Kunugi integrable and Henstock-Kurzweil integrable?.

Definition of the Kunugi integral

Let $\mathcal{E}$ denote the set of all step functions defined on $[a, b]$. We define a neighborhood in $\mathcal{E}$ as follows. Let X be a closed set in $[a, b]$, $\epsilon > 0$, and $f \in \mathcal{E}$. A neighborhood of f, denoted by $V(x, \epsilon; f)$, is the set of all step functions g in $\mathcal{E}$ such that $g = f + r$ and r satisfies the following properties:

(α) $|r(x)| < \epsilon$, for all $x \in X$;

(β) $N|E_N(r)| < \epsilon$, for each N, where $E_N(r) = \{x : |r(x)| > N\}$;

(γ) $|\int_a^b r^N(x)\, dx| < \epsilon$, for each N, where r^N denotes the truncated function of r, i.e., $r^N(x) = r(x)$ when $|r(x)| \leq N$, $r^N(x) = N$ when $r(x) > N$, and $r^N(x) = -N$ when $r(x) < -N$. Here N runs over all positive numbers.

A sequence of neighborhoods $\{V(X_n, \epsilon_n; f_n)\}$ in $\mathcal{E}$ is said to be fundamental if $\{V(X_{n+1}, \epsilon_{n+1}; f_{n+1})\} \subseteq \{V(X_n, \epsilon_n; f_n)\}$ for each n, $\epsilon_n \to 0$ as $n \to \infty$, and each $G_n = [a, b] \setminus X_n$ satisfies the following condition

$$|G_n| < 2^{-n}. \tag{3}$$

For simplicity, we further assume that a fundamental sequence has the properties: $|X_n \setminus X_{n+1}| = 0$ for each n, i.e., X_n is included in X_{n+1} almost everywhere, and ϵ_n converges decreasingly to 0.

Lemma 2

If $\{V_n\}$ *is a fundamental sequence of neighborhoods in* $\mathcal{E}$ *with* $V_n = V(X_n, \epsilon_n; f_n)$, *then*

(i) $\lim_{n\to\infty} f_n(x)$ *exists almost everywhere in* $[a, b]$*;*

(ii) $\lim_{n\to\infty} \int_a^b f_n(x)\ dx$ *exists.*

Definition 3

A function f is said to be Kunugi integrable on $[a, b]$ if there exists a fundamental sequence $\{V(X_n, \epsilon_n; f_n)\}$ in $\mathcal{E}$ such that $\{f_n(x)\}$ converges to $f(x)$ for almost all $x \in [a, b]$ as $n \to \infty$. The integral of f is defined to be

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \int_a^b f_n(x)\ dx.$$

The DL-integral

A function f is said to be DL-integrable on $[a, b]$ if there exists a number B such that for every $\epsilon > 0$ and $\eta > 0$, and for every infinite subset S of positive integers, there exists a positive integer $N \in S$, an open set G, and a constant $\delta > 0$, such that $G \supseteq \{x : |f(x)| > N\}$ and $N|G| < \eta$ and that for every partition $\mathcal{D} = \{[u, v]; \xi\}$ with $0 < v - u < \delta$ and $\xi \in [u, v] \setminus G$, we have

$$|\sum_{\xi \notin G} f(\xi)(v - u) - B| < \epsilon.$$

The set of all DL-integrable functions is a linear space [Ummam and Indrati, 2019].

A function f is Kunugi integrable on $[a, b]$ if and only if f is DL-integrable on $[a, b]$.

The existence of the value of the Kunugi integral on $[a, b]$ is unique. It does not depend on the choice of the fundamental sequence.

The A-integral

A function f is said to be A-integrable on $[a, b]$ if the following conditions are satisfied:

(i) $N|E_N(f)| \to 0$ as $N \to \infty$.

(ii) $\lim_{N\to\infty} \int_a^b f^N(x)\, dx$ exists,

where $E_N(f) = \{x : |f(x)| > N\}$ and $f^N(x) = f(x)$ when $|f(x)| \leq N$, $f^N(x) = N$ when $f(x) > N$, and $f^N(x) = -N$ when $f(x) < -N$, respectively.

The Kunugi integral is equivalent with the A-integral which has many applications to Fourier series.

A function f is DL-integrable on $[a, b]$ if and only if

(i) $N|E_N(f)| \to 0$ as $N \to \infty$.

(ii) $\lim_{N\to\infty} \int_a^b f_N(x)\,dx$ exists,

where $E_N(f) = \{x : |f(x)| > N\}$ and $f_N(x) = f(x)$ when $|f(x)| \leq N$ and $f_N(x) = 0$ when $|f(x)| > N$, respectively.

The integrability of the Henstock-Kurzweil integrability of f does not imply the condition $N|E_N(f)| \to 0$ as $N \to \infty$.

Let consider the function $f : [0,1] \to \mathbb{R}$, in (2), i.e.,

$$f(x) = \begin{cases} n, & x \in (\frac{1}{2}\frac{1}{n+1} + \frac{1}{n}, \frac{1}{n}) \\ \\ -n, & x \in (\frac{1}{n+1}, \frac{1}{2}\frac{1}{n+1} + \frac{1}{n}) \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Here, $N|E_N(f)| \to 1$ as $N \to \infty$.

The integrability of the Henstock-Kurzweil integrability of f does not imply the condition $F_n(a,b) \to F(a,b)$ as $n \to \infty$, ehere $F_n(a,b) = \int_a^b f_n(x)dx$ and $f_n(x) = f(x)$ when $|f(x)| \leq n$ and $f_n(x) = 0$ when $|f(x)| > n$.

Let $n = 1, 2, 3, 4, \ldots$, choose x_n such that

$$\frac{(x_n - \frac{1}{n+1})}{(\frac{1}{n} - x_n)} = \frac{1}{n}.$$

Let define

$$f(x) = \begin{cases} n^2, & x \in (\frac{1}{n+1}, x_n) \\ -n, & x \in (x_n, \frac{1}{n}) \\ 0, & otherwise. \end{cases}$$

The function f is Henstock-Kurzweil integrable to 0 on $[0,1]$. The function $F_n(0,1)$ does not converge to 0 as $n \to \infty$, since

$$F_{n^2}(0,1) = \sum_{i=n+1}^{n^2} \frac{i}{(i+1)^2} \geq \sum_{i=n+1}^{n^2} \frac{1}{i+1}.$$

Some sufficient conditions

Theorem 4

If $f : [a, b] \to \mathbb{R}$ is non-negative measurable function, then f is Henstock-Kurzweil integrable on $[a.b]$ if and only if f is Kunugi integrable on $[a, b]$.

If $f : [a, b] \to \mathbb{R}$ is bounded measurable function, then f is Henstock-Kurzweil integrable on $[a.b]$ if and only if f is Kunugi integrable on $[a, b]$.

Theorem 5

A function f is absolutely Kunugi integrable on $[a, b]$ if and only if f is absolutely Henstock-Kurzweil integrable on $[a, b]$.

Theorem 6

If the function $f : [a, b] \to \mathbb{R}$ *satisfies* $N|E_N(f)| \to 0$ *as* $N \to \infty$ *is Henstock-Kurzweil integrable on* $[a, b]$*, then* f *is Kunugi integrable on* $[a, b]$.

If the function $f : [a, b] \to \mathbb{R}$ *satisfies* $\int_a^x f_n(t)dt$ *converges for every* $x \in [a, b]$*, then* f *is Henstock-Kurzweil integrable on* $[a, b]$.

Conclusion

The condition in the last theorem in having the sufficient condition of the Kunugi integrable function to be Henstock-Kurcweil integrable is actually a Cauchy principal in the integral.

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